

DISCRETE SINGULAR INTEGRALS IN A HALF-SPACE

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ABSTRACT. We consider Calderon – Zygmund singular integral in the discrete half-space $h\mathbf{Z}_+^m$, where \mathbf{Z}^m is entire lattice ($h > 0$) in \mathbf{R}^m , and prove that the discrete singular integral operator is invertible in $L_2(h\mathbf{Z}_+^m)$ iff such is its continual analogue. The key point for this consideration takes solvability theory of so-called periodic Riemann boundary problem, which is constructed by authors.

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1. INTRODUCTION

We consider simplest Calderon-Zygmund operators of convolution type [1]

$$v.p. \int_{\mathbf{R}^m} K(x-y)u(y)dy = \lim_{\substack{\varepsilon \rightarrow 0 \\ N \rightarrow +\infty}} \int_{\varepsilon < |x-y| < N} K(x-y)u(y)dy,$$

where the kernel $K(x)$ satisfies the following conditions:

- 1) $K(tx) = t^{-m}K(x), \forall x \neq 0, t > 0$;
- 2) $\int_{S^{m-1}} K(\theta)d\theta = 0, S^{m-1}$ is the unit sphere in \mathbf{R}^m ;
- 3) $K(x)$ is differentiable on $\mathbf{R}^m \setminus \{0\}$.

Let us consider a discrete operator generated by the Calderon-Zygmund kernel $K(x)$, and defined on functions $u_h(\tilde{x})$, $\tilde{x} \in h\mathbf{Z}^m$, where \mathbf{Z}^m is entire lattice ($h > 0$) in \mathbf{R}^m , and the corresponding equation

$$(1) \quad au_h(\tilde{x}) + \sum_{\tilde{y} \in h\mathbf{Z}_+^m} K(\tilde{x} - \tilde{y})u_h(\tilde{y})h^m = v_h(\tilde{x}), \quad \tilde{x} \in h\mathbf{Z}_+^m,$$

a is certain constant, in the discrete half-space $h\mathbf{Z}_+^m = \{\tilde{x} \in h\mathbf{Z}^m : \tilde{x}_m > 0\}$, $u_h, v_h \in L_2(h\mathbf{Z}_+^m)$.

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By definition we put $K(0) = 0$, and for the operator

$$u_h(\tilde{x}) \mapsto au(\tilde{x}) + \sum_{\tilde{y} \in h\mathbf{Z}^m} K(\tilde{x} - \tilde{y})u_h(\tilde{y})h^m, \quad \tilde{x} \in h\mathbf{Z}^m,$$

we introduce its symbol by the formula

$$\sigma_h(\xi) = a + \sum_{\tilde{x} \in h\mathbf{Z}^m} e^{-i\xi\tilde{x}} K(\tilde{x})h^m;$$

it is periodic function with basic cube period $[-\pi h^{-1}; \pi h^{-1}]^m$.

The sum for $\sigma_h(\xi)$ is defined as a limit of partial sums over cubes Q_N

$$\lim_{N \rightarrow \infty} \sum_{\tilde{x} \in Q_N} e^{-i\xi\tilde{x}} K(\tilde{x})h^m,$$

$$Q_N = \left\{ \tilde{x} \in h\mathbf{Z}^m : |\tilde{x}| \leq N, |\tilde{x}| = \max_{1 \leq k \leq m} |\tilde{x}_k| \right\}.$$

It is very similar classical symbol of Calderon-Zygmund operator [1], which is defined as Fourier transform of the kernel $K(x)$ in principal value sense

$$\sigma(\xi) = \lim_{\substack{N \rightarrow \infty \\ \varepsilon \rightarrow 0}} \int_{\varepsilon < |x| < N} K(x) e^{i\xi x} dx.$$

Key point of our study is a theorem proved in [6], asserting that images of σ and σ_h are the same.

We also introduce continual equation in a half-space

$$(2) \quad au(x) + \int_{\mathbf{R}_+^m} K(x - y)u(y)dy = v(x), \quad x \in \mathbf{R}_+^m,$$

and we'll prove that the equations (1) and (2) are uniquely solvable or unsolvable simultaneously for all $h > 0$ in corresponding spaces.

2. DISCRETE CALDERON-ZYGMUND OPERATORS

2.1. Symbol properties. We recall some properties of symbols $\sigma(\xi)$ and $\sigma_h(\xi)$, which are needed for us [6].

Lemma 1. $\lim_{h \rightarrow 0} \sigma_h(\xi) = \sigma(\xi), \quad \forall \xi \neq 0.$

Proof. Indeed, if we fix $\xi \neq 0$, then by definition of integral as a limit of integral sums, we finish the proof. \square

Lemma 2. $\sigma_h(\xi) = \sigma_1(h\xi), \quad \forall h > 0, \quad \xi \in [-\pi h^{-1}, \pi h^{-1}]^m.$

Proof.

$$\begin{aligned}\sigma_h(\xi) &= \sum_{\tilde{x} \in h\mathbf{Z}^m} K(\tilde{x})e^{-i\tilde{x} \cdot \xi} h^m = \\ &= \sum_{\tilde{y} \in \mathbf{Z}^m} K(h\tilde{y})e^{-i\tilde{y} \cdot h\xi} h^m = \sum_{\tilde{y} \in \mathbf{Z}^m} K(\tilde{y})e^{-i\tilde{y} \cdot h\xi} = \sigma_1(h\xi).\end{aligned}$$

□

Lemma 3. *The images of σ and σ_h are the same, and their values are constant for any ray from origin.*

Proof. It follows from previous lemmas immediately, because if we fix ξ , then $\sigma_1(0) = \sigma(\xi) \implies \sigma_h(0) = \sigma(\xi)$. □

2.2. Symbols and operators. We consider more general case in the whole space \mathbf{R}^m

$$(M_1P_+ + M_2P_-)U = V,$$

taking into account that M_1, M_2 are operators of type (2), and P_+, P_- are restriction operators on $\mathbf{R}_\pm^m = \{x = (x_1, \dots, x_m), \pm x_m > 0\}$. It is easily verified that the equation (2) is a special case for such equation, when $M_2 \equiv I$, I is identity operator.

If we'll denote the Fourier transform by letter F , and use the notations [4]

$$\begin{aligned}FP_+ &= Q_{\xi'}F, \quad FP_- = P_{\xi'}F, \\ P_{\xi'} &= 1/2(I + H_{\xi'}), \quad Q_{\xi'} = 1/2(I - H_{\xi'}),\end{aligned}$$

where $H_{\xi'}$ is Hilbert transform on variable ξ_m for fixed $\xi' = (\xi_1, \dots, \xi_{m-1})$ [5]:

$$(H_{\xi'}u)(\xi', \xi_m) \equiv \frac{1}{\pi i} v.p. \int_{-\infty}^{+\infty} \frac{u(\xi', \tau)}{\tau - \xi_m} d\tau,$$

then the equation mentioned after applying the Fourier transform will be the following equation with the parameter ξ' :

$$\begin{aligned}&\frac{\sigma_{M_1}(\xi', \xi_m) + \sigma_{M_2}(\xi', \xi_m)}{2} \tilde{U}(\xi) + \\ &+ \frac{\sigma_{M_1}(\xi', \xi_m) - \sigma_{M_2}(\xi', \xi_m)}{2\pi i} v.p. \int_{-\infty}^{+\infty} \frac{\tilde{U}(\xi', \eta)}{\eta - \xi_m} d\eta = \tilde{V}(\xi)\end{aligned}$$

(\sim denotes the Fourier transform).

This equation is closely related to boundary Riemann problem with the parameter ξ' with coefficient [2, 3]

$$G(\xi', \xi_m) = \sigma_{M_1}(\xi', \xi_m) \sigma_{M_2}^{-1}(\xi', \xi_m).$$

3. PERIODIC RIEMANN BOUNDARY PROBLEM

The theory of periodic Riemann boundary problem was constructed by authors [7] (see also forthcoming paper with the same name in *Differential Equations*) with full details, and now we will use its general consequences.

Let's denote $\mathbf{Z}_+ = 0, 1, 2, \dots$, $\mathbf{Z}_- = \mathbf{R} \setminus \mathbf{Z}_+$. The Fourier transform for function of discrete variable is the series

$$(3) \quad (Fu)(\xi) = \sum_{k=-\infty}^{+\infty} u(k)e^{-ik\xi}, \quad \xi \in [-\pi, \pi].$$

Let's consider the Fourier transform (3) for the indicator of \mathbf{Z}_+ :

$$\chi_{\mathbf{Z}_+}(x) = \begin{cases} 1, & x \in \mathbf{Z}_+ \\ 0, & x \notin \mathbf{Z}_+ \end{cases}.$$

For summable functions their product transforms to convolution of their Fourier images on the segment $[-\pi, \pi]$ but for our case $F(\chi_{\mathbf{Z}_+} \cdot u)$ one of functions $\chi_{\mathbf{Z}_+}$ is not summable. Thus, first we introduce some regularizing multiplier and evaluate the following Fourier transform

$$\begin{aligned} F(e^{-\tau k} \cdot \chi_{\mathbf{Z}_+})(\xi) &= \frac{1}{2\pi} \sum_{k \in \mathbf{Z}_+} e^{-\tau k} e^{-ik\xi} = \frac{1}{2\pi} \sum_{k \in \mathbf{Z}_+} e^{-\tau k - ik\xi} = \\ &= \frac{1}{2\pi} \sum_{k \in \mathbf{Z}_+} e^{-ik(\xi + i\tau)} = \frac{1}{2\pi} \sum_{k \in \mathbf{Z}_+} e^{-ikz}, \quad \tau \rightarrow 0, \quad z = \xi + i\tau, \quad \tau > 0. \end{aligned}$$

The Fourier transform for the function $u(n)$ we'll denote $\hat{u}(\xi)$, it is left to find the sum for e^{-ikz} ,

$$\frac{1}{2\pi} \sum_{k \in \mathbf{Z}_+} e^{-ikz} = \frac{1}{2\pi} (1 + e^{-iz} + e^{-2iz} + \dots) = \frac{1}{2\pi} \frac{1}{1 - e^{-iz}},$$

After some transformations:

$$F(\chi_{\mathbf{Z}_+} \cdot u)(\xi) = \lim_{\tau \rightarrow 0+} \left(\frac{\hat{u}(\xi)}{4\pi} + \frac{1}{4\pi i} \int_{-\pi}^{\pi} \hat{u}(t) \cot \frac{z-t}{2} dt \right), \quad z = \xi + i\tau.$$

According to Sokhotskii formulas (these are almost same for periodic kernel $\cot(x)$) (see also classical books [2, 3])

$$F(\chi_{\mathbf{z}_+} \cdot u)(\xi) = \frac{\hat{u}(\xi)}{4\pi} + \frac{1}{4\pi i} \int_{-\pi}^{\pi} \hat{u}(t) \cot \frac{\xi - t}{2} dt + \frac{\hat{u}(\xi)}{2}$$

If we introduce the function $\chi_{\mathbf{z}_-}(x)$ and consider the Fourier transform for the product $F(\chi_{\mathbf{z}_-} \cdot u)$ with preliminary regularization, then we have

$$\begin{aligned} F(e^{-\tau k} \cdot \chi_{\mathbf{z}_-}) &= \frac{1}{2\pi} \sum_{-\infty}^{-1} e^{\tau k} e^{-ik\xi} = \frac{1}{2\pi} \sum_{-\infty}^{-1} e^{\tau k - ik\xi} = \\ &= \frac{1}{2\pi} \sum_{-\infty}^{-1} e^{-ik(\xi + i\tau)} = \frac{1}{2\pi} \sum_{-\infty}^{-1} e^{-ikz}, \quad \tau \rightarrow 0, \quad z = \xi + i\tau, \quad \tau < 0. \end{aligned}$$

Further,

$$\frac{1}{2\pi} \sum_{-\infty}^{-1} e^{-ikz} = \frac{1}{2\pi} (-1 + 1 + e^{iz} + e^{2iz} + \dots) = -\frac{1}{2\pi} + \frac{1}{2\pi} \frac{1}{1 - e^{iz}},$$

With the help of some elementary calculations:

$$F(\chi_{\mathbf{z}_-} \cdot u) = \lim_{n \rightarrow \xi, \tau \rightarrow 0} \left(-\frac{\hat{u}(\xi)}{4\pi} - \frac{1}{4\pi i} \int_{-\pi}^{\pi} \hat{u}(t) \cot \frac{z - t}{2} dt \right), \quad z = \xi + i\tau.$$

Applying Sokhotskii formulas, we have:

$$F(\chi_{\mathbf{z}_-} \cdot u)(\xi) = \frac{\hat{u}(\xi)}{4\pi} + \frac{1}{4\pi i} \int_{-\pi}^{\pi} \hat{u}(t) \cot \frac{\xi - t}{2} dt + \frac{\hat{u}(\xi)}{2}.$$

To verify one can find the sum for $F(\chi_{\mathbf{z}_+} \cdot u)$, $F(\chi_{\mathbf{z}_-} \cdot u)$ and obtain:

$$\begin{aligned} F(\chi_{\mathbf{z}_+} \cdot u) + F(\chi_{\mathbf{z}_-} \cdot u) &= \frac{\hat{u}(\xi)}{4\pi} + \frac{1}{4\pi i} \int_{-\pi}^{\pi} \hat{u}(t) \cot \frac{\xi - t}{2} dt + \frac{\hat{u}(\xi)}{2} - \\ &\quad - \frac{\hat{u}(\xi)}{4\pi} - \frac{1}{4\pi i} \int_{-\pi}^{\pi} \hat{u}(t) \cot \frac{\xi - t}{2} dt + \frac{\hat{u}(\xi)}{2} = \hat{u}(\xi). \end{aligned}$$

These calculations lead to certain periodic Riemann boundary value problem, for which the solvability conditions are defined by the index of its coefficient. The problem is formulated as following way: finding

two functions $\Phi^\pm(t)$ which admit an analytical continuation into upper and lower half-strip in the complex plane \mathbf{C} , real part is the segment $[-\pi, \pi]$, and their boundary values satisfy the relation

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in [-\pi, \pi],$$

$G(t), g(t)$ are given functions on $[-\pi, \pi]$, and such that $G(-\pi) = G(\pi), g(-\pi) = g(\pi)$. Index for such problem is called the integer number

$$\varkappa = \frac{1}{2\pi} \int_{-\pi}^{\pi} d \arg G(t).$$

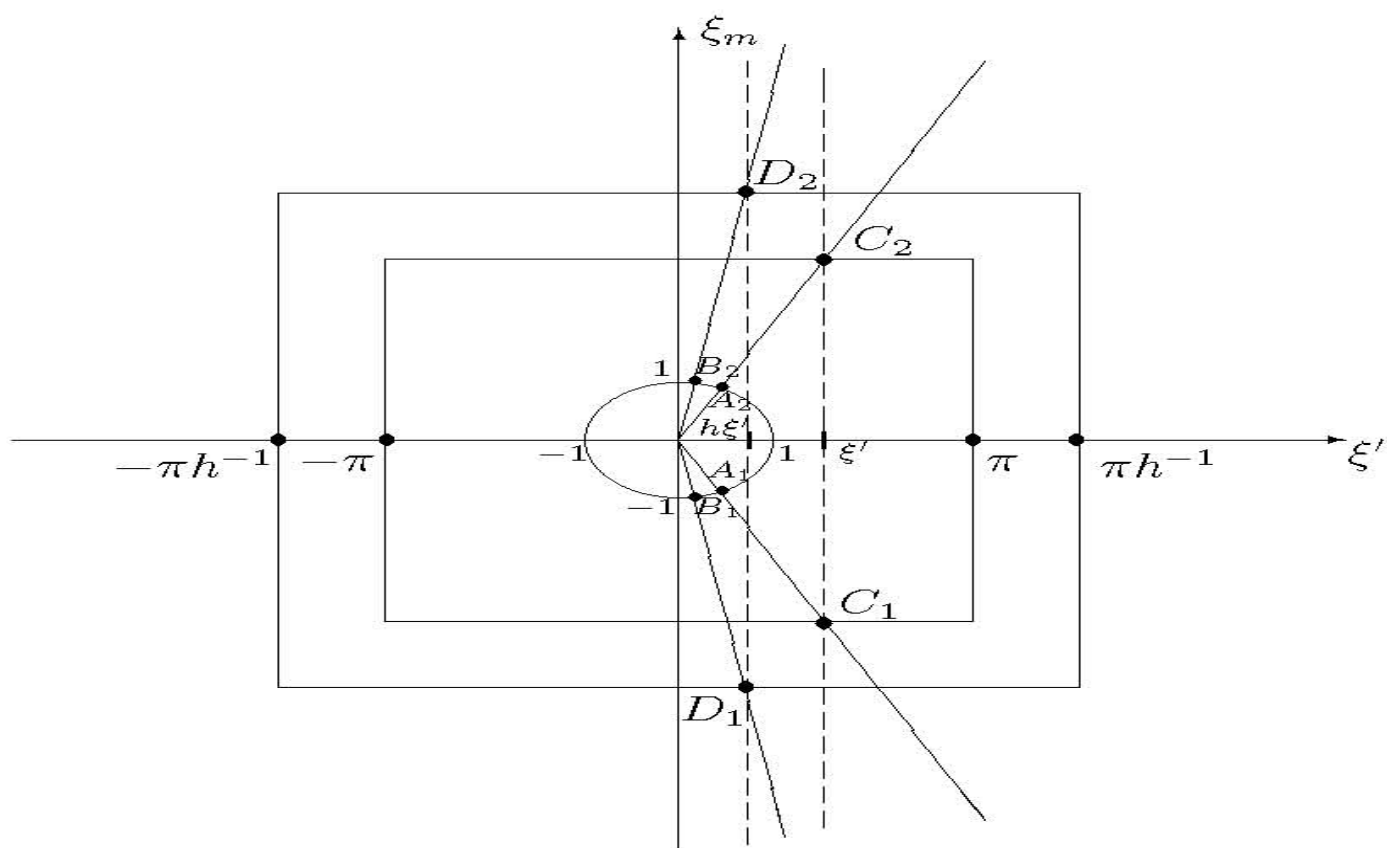


Fig. 1. Varying the symbol

4. SOLVABILITY CONDITIONS

Here we suppose additionally, that the symbol $\sigma(\xi', \xi_m)$ satisfies the condition

$$\sigma(0, \dots, 0, -1) = \sigma(0, \dots, 0, +1).$$

Main Theorem. *The equations (1) and (2) are uniquely solvable or unsolvable simultaneously for all $h > 0$.*

Proof. We need to look our symbols $\sigma(\xi)$ and $\sigma_h(\xi)$ more exactly. We'll illustrate our consideration with the help of Fig.1.

If we fix ξ' in the cube $[-\pi, \pi]^m$, then under varying ξ_m on $[-\pi, \pi]$ the argument of $\sigma_1(\xi)$ will vary along the curve on a cubical surface of $[-\pi, \pi]^m$, which unites the points C_1 and C_2 (for the case $m \geq 3$ all such curves are homotopic, and for the case $m = 2$ there are two curves left and right one).

This varying corresponds to the varying of the argument of function $\sigma(\xi)$ along the curve from point A_1 to point A_2 on the unit sphere. Further, if we consider the symbol $\sigma_h(\xi)$ now on the cube $[-h^{-1}\pi, h^{-1}\pi]^m$, then according to lemma 2.2 $h\xi_m$ will be varied on $[-\pi, \pi]$ also under fixed $h\xi'$. In other words, the argument of $\sigma_h(\xi)$ for fixed ξ' (we consider small $h > 0$) will be varied along a curve on a cubical surface of $[-h^{-1}\pi, h^{-1}\pi]^m$, which unites the points D_1 and D_2 . It corresponds to varying argument of function $\sigma(\xi)$ from point B_1 to point B_2 on the unit sphere. Obviously, under decreasing h the sequence A_1, B_1, \dots will be convergent to the south pole of the unit sphere $(0, \dots, 0, -1)$, and the sequence A_2, B_2, \dots to the north pole $(0, \dots, 0, +1)$. Thus, because the variation of an argument of $\sigma_h(\xi)$ on ξ_m under fixed ξ' is $2\pi k$ (σ_h is periodic function), then under additional assumption

$$\sigma(0, \dots, 0, -1) = \sigma(0, \dots, 0, +1)$$

(this property is usually called the transmission property) we'll obtain that variation of an argument for the function $\sigma(\xi)$ under varying ξ_m from $-\infty$ to $+\infty$ under fixed ξ' (this variation of $\sigma(\xi)$ moves along the arc of a big half-circumference on the unit sphere) is also $2\pi k$. According to our assumptions on continuity of $\sigma(\xi)$ on the unit sphere, it will be the same number $2\pi k, k \in \mathbf{Z}$,

$$\lim_{h \rightarrow 0} \int_{-\pi h^{-1}}^{\pi h^{-1}} d \arg \sigma_h(\xi', \xi_m) = \int_{-\infty}^{+\infty} d \arg \sigma(\xi', \xi_m), \quad \forall \xi' \neq 0.$$

So, both for the equation (1) and equation (2) the uniquely solvability condition is defined by the same number. This completes the proof. \square

5. CONCLUSION

We see that both continual and discrete equations are solvable or unsolvable simultaneously, and then we need to find good finite approximation for infinite system of linear algebraic equations for computer calculations. First steps in this direction were done in the paper [8], where the authors suggested to use fast Fourier transform.

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